

# A NOTE ON PROJECTIVIZED TANGENT CONE QUADRICS OF RANK $\leq 4$ IN THE IDEAL OF A PRYM-CANONICAL CURVE

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**ABSTRACT.** Throughout the paper, among other results, we give in theorem 3.1 and proposition 3.2 a partial analogue of theorem 1.1 for projectivized tangent cone quadrics of rank equal or less than 4, for Prymians. During the lines of the paper it would be seen that for an un-ramified double covering of a general smooth tetragonal curve  $X$  induced by a line bundle  $\eta$  on  $X$  with  $\eta^2 = 0$ , the Prym-canonical model of  $X$  is projectively normal in  $\mathbf{P}(H^0(K_X \cdot \eta))$ . Then we consider a genus  $g = 7$ , tetragonal curve  $C$  which is birationally isomorphic to a plane sextic curve  $X$  with ordinary singularities. As byproduct of theorem 3.1 and proposition 3.2, we show that the stable projectivized tangent cone quadrics with rank equal or less than 4 of an un-ramified double covering of  $C$ , generate the space of quadrics in  $\mathbf{P}(H^0(K_C \cdot \eta))$  containing  $K_C \cdot \eta$ -model of  $C$ , where  $\eta$  is a line bundle on  $C$  with  $\eta^2 = 0$ , obtained in section 4.

**Keywords:** Clifford Index; Projectivized Tangent Cone; Prym-Canonical Curve; Prym variety; Tetragonal Curve.

**MSC(2010):** 14H99; 14C20; 14H50; 14H40; 14H51.

## 1. INTRODUCTION

For an étale double covering  $\pi : \tilde{X} \rightarrow X$  of smooth curves, it is naturally associated a principally polarized abelian variety the so called Prym variety of  $\pi$ , which is denoted by  $\mathbb{P}(\pi)$ , whose principal polarization is induced twice by  $\Theta_{\tilde{X}}$ , the theta divisor of  $\tilde{X}$ . While this P.P.A.V. enjoys from some interesting properties analogous to Jacobians, it behaves differently in some another properties. Usually in most cases these differences lead to a rich geometry which provides wide areas of research. For example, although the theory of Prym varieties is old enough and has been studied variously by many well known mathematicians since decades ago, but surprisingly an analogue of the well known Riemann singularity theorem for Prymians has been given

relatively lately, by R. Smith and R. Varley in [13] and its complete analogue has given recently by S. C. Martin in [12].

Another useful and nice package in the land of Jacobians of canonical curves, is the well known theorem 1.1, proved by Andreotti-Mayer in [2] and by G. Kempf in [10].

**Theorem 1.1.** *Let  $X$  be a smooth projective curve of genus  $g$  on an algebraically closed field of characteristic zero and  $|D| = g_{g-1}^1$  a complete linear series of degree  $g-1$  and dimension 1 on  $X$ . Consider the corresponding double point of  $\Theta_X$ :*

$$\mathcal{O}_X(D) = \mathcal{O}_X(K_X - D) \in \Theta_{\text{sing}}.$$

*Then the projectivized tangent cone to  $\Theta_X$  at  $\mathcal{O}_X(D)$  is a quadric of rank at least or equal to 4 containing the canonical model of  $X$  which can be described as the union of the linear span of divisors in  $|D| = g_{g-1}^1$ . Moreover the quadric is of rank 3 precisely when  $|2D| = |K_X|$ . Conversely a quadric  $Q$  of rank less than or equal to 4, through  $X$  is a tangent cone to  $\Theta_X$  if one of its rulings cuts out a complete linear series of degree  $g-1$  and dimension 1 on  $X$ .*

Although it is completely known in the literature that the projectivized tangent cone at a double point  $a$  of Prym-Theta divisor of general Prym-Canonical curves is a quadric of rank 6 rather than rank 4, but it might be interesting to know:

- How is the effect of linear subspaces of a Prym quadric tangent cone on  $C$ , when the quadric is of rank 4 containing  $C$ ?

Equivalently we look for an analogue of theorem 1.1 for Prymians. The genus  $g = 7$  case, the first case where the singular locus of prym theta divisor is nonempty, is the first case that has to be dealt with. We see that a projectivized tangent cone quadric of rank equal or less than 4 at a stable singularity of Prym-theta divisor of an étale double covering  $\tilde{X} \rightarrow X$ , through the Prym-canonical model of  $X$ , imposes a linear series of degree  $d$  such that  $d \in \{g-3, g-2, g-1\}$ . If  $d \in \{g-2, g-1\}$  then for each  $g \geq 7$  the linear series is complete, while it would be complete for  $d = g-3$  too provided that  $g = 7$ . A partial converse to this result will be given in Proposition 3.2.

Then we proceed to provide an evidence for the above question. Precisely we give an example of a curve admitting projectivized tangent cone quadrics of rank equal or less than 4. We would like to give an example, on which not only a complete converse of theorem 3.1 is valid, but also its rank 4 Prym quadric tangent cones generate the space of quadrics containing Prym-canonical model of  $C$ . A curve which is

birationally equivalent to a plane sextic with three number of ordinary singularities, might seem a candidate for this aim. But because of technical reasons a curve of this type with three collinear ordinary singularities is needed. Whereas the existence of a plane sextic curve with three number of ordinary double points is a well known fact, the existence of a plane sextic curve with three number of collinear ordinary singularities needs an actual proof. Such a proof will be given in Theorem 4.1.

H. Lange and E. Sernesi in [11] have proved that any Prym-canonical line bundle on a curve of Clifford index  $\geq 3$  is globally generated and very ample and moreover its Prym-canonical model is projectively normal in the projective space of the Prym-canonical differential forms. When this is no longer true for an arbitrary tetragonal curve, it would be verified not only for Prym-canonical model of the example, obtained in Theorem 4.1, in the projective space of its Prym-canonical differential forms. but also for Prym-canonical model of general tetragonal curves in their projective space of their Prym-canonical differential forms. See Lemma 4.2.

## 2. PRELIMINARIES AND NOTATIONS

A nontrivial line bundle  $\eta$  on an irreducible nonsingular projective curve  $X$  with  $\eta^2 = 0$  gives rise to a double covering  $\pi : \tilde{X} \rightarrow X$  and vice versa. For a nontrivial line bundle  $\eta$  with  $\eta^2 = 0$  we denote by  $\pi_\eta$  the map induced by  $\eta$ . The kernel of the norm map of  $\pi_\eta$  denoted by  $\text{Nm}(\pi_\eta)$ , which is a subset of  $J(\tilde{X})$ , turns to be the union of two irreducible isomorphic components one of them containing zero. The component containing zero, denoted by  $\mathbb{P}(\pi_\eta)$ , is called the Prym variety of the double covering  $\pi_\eta$  and consists of line bundles  $\tilde{\mathcal{H}}$  on  $\tilde{X}$ , with  $\tilde{\mathcal{H}} \in \text{Ker}(\text{Nm})$  and  $h^0(\tilde{\mathcal{H}})$  is an even number. The other one which is denoted by  $Z_1$ , consists of line bundles  $\tilde{\mathcal{H}}$  on  $\tilde{X}$ , with  $\tilde{\mathcal{H}} \in \text{Ker}(\text{Nm})$  and  $h^0(\tilde{\mathcal{H}})$  is an odd number.

The theta divisor  $\Theta_{\tilde{X}}$  induces a principal polarization on  $\mathbb{P}(\pi_\eta)$  and the Prym variety  $\mathbb{P}(\pi_\eta)$  turns to be a principally polarized abelian variety with principal polarization  $\mathbb{E}(\pi_\eta)$ . In terms of dimensions of global sections of the points in  $J(\tilde{X})$ , the principally polarized abelian variety  $\mathbb{P}(\pi_\eta)$  can be described as follows:

$$\mathbb{P}(\pi_\eta) = \{\tilde{\mathcal{L}} \in J(\tilde{X}) \mid \text{Nm}(\tilde{\mathcal{L}}) = K_X, h^0(\tilde{\mathcal{L}}) \equiv 0 \pmod{2}\}.$$

The singular locus of  $\mathbb{E}(\pi_\eta)$  has a similar description in these terms too:

$$\begin{aligned} \text{Sing}(\mathbb{E}(\pi_\eta)) &= \{\tilde{\mathcal{L}} \in \mathbb{P}(\pi_\eta) \mid h^0(\tilde{\mathcal{L}}) \geq 4\} \\ &\cup \{\tilde{\mathcal{L}} \in \mathbb{P}(\pi_\eta) \mid h^0(\tilde{\mathcal{L}}) = 2, T_{\tilde{\mathcal{L}}}(\mathbb{P}(\pi_\eta)) \subset TC_{\tilde{\mathcal{L}}}(\Theta_{\tilde{X}})\} \end{aligned}$$

where  $TC_{\tilde{\mathcal{L}}}(\Theta_{\tilde{X}})$  denotes the tangent cone of  $\Theta_{\tilde{X}}$  at  $\tilde{\mathcal{L}}$ . The singular points of  $\mathbb{E}(\pi_\eta)$  with  $h^0(\tilde{\mathcal{L}}) \geq 4$  are called stable singularities and singularities belonging to the second set are called exceptional ones. For standard notations and details of the subject, see of [5, Chapter 14].

If  $\eta$  is a nontrivial line bundle on  $X$  such that  $\eta^2 = 0$ , then the line bundle  $K_X \cdot \eta$  is called a Prym-canonical line bundle on  $X$ , if  $K_X \cdot \eta$  is globally generated and very ample, the irreducible (possibly singular) curve  $\phi_{K_X \cdot \eta}(X)$  is a Prym-canonical curve, where

$$\phi_{K_X \cdot \eta} : X \rightarrow \mathbf{P}(H^0(K_X \cdot \eta))$$

is the morphism defined by global sections of  $K_X \cdot \eta$ . We will denote the curve  $\phi_{K_X \cdot \eta}(X)$  by  $X_\eta$ . A linear series  $g_d^r$  on  $X$  gives rise to a same linear series on  $X_\eta$  via  $\phi_{K_X \cdot \eta}$  and vice versa. In the absence of any confusion, we use a same symbol for both of these linear series on  $X$  or on  $X_\eta$ .

**Theorem 2.1.** *Let  $\pi : \tilde{X} \rightarrow X$  be an étale double covering induced by a line bundle  $\eta$  such that  $\eta^2 = 0$ . Assume moreover that the line bundle  $K_X \cdot \eta$  is very ample and globally generated. Then the projectivized tangent cone of  $\mathbb{E}(\pi_\eta)$  at a double point  $\tilde{\mathcal{L}}$  is a quadric hypersurface in  $\mathbf{P}(H^0(K_X \cdot \eta))$  containing  $X_\eta$  if and only if  $\tilde{\mathcal{L}}$  is a stable singularity with  $h^0(\tilde{\mathcal{L}}) = 4$ .*

*Proof.* Let  $\tilde{\mathcal{L}} \in \text{Sing}(\mathbb{E}(\pi_\eta))$  be a double point of  $\mathbb{E}(\pi_\eta)$ . Consider that using [12, Corollary 6.2.5], we have  $\phi_{K_X \cdot \eta}(X) \subset \mathbb{P}TC_{\tilde{\mathcal{L}}}(\mathbb{E}(\pi_\eta))$  if and only if  $h^0(\tilde{\mathcal{L}}) \geq 4$ . If  $h^0(\tilde{\mathcal{L}}) > 4$  then one would have  $h^0(\tilde{\mathcal{L}}) \geq 6$ . Therefore by Riemann-Kempf singularity theorem  $\deg(\mathbb{P}TC_{\tilde{\mathcal{L}}}(\Theta_{\tilde{X}})) \geq 6$ . Now since  $\mathbb{P}TC_{\tilde{\mathcal{L}}}(\mathbb{E}(\pi_\eta)) = 2\mathbb{P}TC_{\tilde{\mathcal{L}}}(\Theta_{\tilde{X}}) \cdot \mathbf{P}(H^0(K_X \cdot \eta))$ , the hypersurface  $\mathbb{P}TC_{\tilde{\mathcal{L}}}(\mathbb{E}(\pi_\eta))$  would be of degree at least 3 and vice versa.  $\square$

**Lemma 2.2.** *If  $\tilde{p}, \tilde{q} \in \tilde{X}$  and  $L_{\tilde{p}, \tilde{q}}$  is the line in  $\mathbf{P}(H^0(K_{\tilde{X}}))$  joining  $\tilde{p}$  to  $\tilde{q}$  then  $\phi_{K_X \cdot \eta}(\pi(\tilde{p})) = \bar{p} = L_{\tilde{p}, \tilde{q}} \cap \mathbf{P}(H^0(K_X \cdot \eta))$ .*

*Proof.* This is claimed and proved in [14, page 4954].  $\square$

### 3. RANK $\leq 4$ PROJECTIVIZED TANGENT CONE QUADRICS

Assume that  $F$  is a smooth projective curve of genus  $g$  with a very ample Prym-canonical line bundle  $K_F \cdot \eta$ .

**Theorem 3.1.** *Assume that  $F$  is a smooth non-hyperelliptic projective curve of genus  $g$  with a very ample Prym-canonical line bundle  $K_F \cdot \eta$ . Assume moreover that  $\pi_\eta : \tilde{F} \rightarrow F$  is an étale double cover of  $F$ . Let  $Q$  be a quadric of rank equal or less than 4 containing  $F_\eta$  in  $\mathbf{P}(H^0(K_F \cdot \eta))$ .*

Furthermore for  $\tilde{\mathcal{L}} \in \text{Sing}(\mathbb{E}(\pi_\eta))$ , assume that  $Q$  is the projectivized tangent cone of  $\mathbb{E}(\pi_\eta)$  at  $\tilde{\mathcal{L}}$ . Then one of the rulings of  $Q$  cuts a  $g_d^1$  on  $F_\eta$  with  $g-3 \leq d \leq g-1$  and  $2\mathcal{O}(g_d^1) \otimes \mathcal{L} = K_F$ , for some line bundle  $\mathcal{L}$  on  $F$  which  $\mathcal{L}$  is of degree 0, 2 or 4. Additionally, the  $g_d^1$  is complete when  $d \in \{g-2, g-1\}$ . It is complete in the case  $d = g-3$  as well, provided that  $g = 7$ .

*Proof.* Assume that  $\pi_\eta : \tilde{F} \rightarrow F$  is an étale double covering. If  $Q = Q_{\tilde{\mathcal{L}}}$  is a projectivized tangent cone of  $\text{Sing}(\mathbb{E}(\pi_\eta))$  at  $\tilde{\mathcal{L}}$  which is a quadric of rank equal or less than 4 containing  $F_\eta$ , then one of its rulings cuts a  $g_d^1$ , on  $F_\eta$ . For a divisor  $E \in |g_d^1|$ , using the geometric Riemann-Roth theorem and considering that the linear space  $\langle E \rangle$  inside  $Q_{\tilde{\mathcal{L}}}$  is of codimension 2 in  $\mathbf{P}(H^0(K_F \cdot \eta))$ , one obtains that  $h^0(K_F \cdot \eta - E) = 2$ . Since  $\dim(\langle E \rangle) = g-4$  one has  $d = \deg(E) \geq g-3$ .

Set  $\Lambda = \langle E \rangle$  and consider that  $\Lambda = \tilde{\Lambda} \cap \mathbf{P}(H^0(K_F \cdot \eta))$  where  $\tilde{\Lambda} = \langle \tilde{E} \rangle$  for some  $\tilde{E} \in \tilde{F}_{2g-2}^3$  such that  $\tilde{\mathcal{L}} = \mathcal{O}(\tilde{E})$ , see [13]. For  $p \in F_\eta$  setting  $\pi_\eta^{-1}(p) = \{\tilde{p}, \tilde{q}\}$  using Lemma 2.2, we have  $p \in \text{Supp}(E)$  if and only if  $\{\tilde{p}, \tilde{q}\} \subset \text{Supp}(\tilde{E})$ . This observation implies that  $d = \deg(E) \leq g-1$ .

To see the completeness of  $g_d^1$  assume that  $g_d^1 \subset |D|$  for some divisor  $D$  in  $g_d^1$  and set  $\tilde{D} = \pi_\eta^*(D)$ .

Assume first that  $d = g-1$ : then the equality  $h^0(D \cdot \eta) = h^0(K_F \cdot \eta - D)$  together with the geometric Riemann-Roth theorem imply that  $h^0(D \cdot \eta) = 2$ . The assumption  $d = g-1$ , implies that  $2g_d^1 = K_{F_\eta}$  and one can see from this that  $\pi_\eta^*(g_d^1) \in \text{Sing}(\mathbb{E}(\pi_\eta))$ . In fact by Lemma 2.2, for each divisor  $\Gamma \in |g_d^1|$  the divisor  $\tilde{\Gamma} = \pi_\eta^*(\Gamma)$  is a divisor associated to a global section of  $\tilde{\mathcal{L}}$ , so one has  $\pi_\eta^*(g_d^1) = \tilde{\mathcal{L}}$ , in this case. Therefore  $\pi_\eta^*(g_d^1) \in \text{Sing}(\mathbb{E}(\pi_\eta))$ .

If  $h^0(g_d^1) > 2$  then  $h^0(\pi_\eta^*(g_d^1)) = h^0(g_d^1) + h^0(g_d^1 \cdot \eta) > 4$ . This implies that  $Q_{\pi_\eta^*(g_d^1)} = \cup_{\tilde{D} \in |\pi_\eta^*(g_d^1)|} \langle \tilde{D} \rangle$  is a hypersurface at least of degree 6 in  $\mathbf{P}(H^0(K_{\tilde{F}}))$ . Therefore the hypersurface  $Q = \frac{1}{2}[(Q_{\pi_\eta^*(g_d^1)}) \cdot \mathbf{P}(H^0(K_F \cdot \eta))]$  would be a hypersurface of degree at least three. This by equality of the quadrics  $Q$  and  $Q_{\tilde{\mathcal{L}}}$  is absurd. This implies that the  $g_{g-1}^1$  is complete.

If  $d = g-2$  then  $h^0(D \cdot \eta) = 1$  and there exist  $\tilde{p}, \tilde{q} \in \tilde{F}$  such that  $\mathcal{O}(\tilde{D}) \otimes \mathcal{O}(\tilde{p} + \tilde{q}) \in \text{Sing}(\mathbb{E}(\pi_\eta))$ . In fact as in the previous case for each divisor  $\Gamma \in |g_d^1|$  there are points  $\tilde{p}, \tilde{q} \in \tilde{F}$  such that  $\tilde{\Gamma} = \pi_\eta^*(\Gamma) + (\tilde{p} + \tilde{q})$  is a divisor associated to a global section of  $\tilde{\mathcal{L}}$ . So one has

$$|\mathcal{O}(\tilde{D}) \otimes \mathcal{O}(\tilde{p} + \tilde{q})| = |\tilde{\mathcal{L}}| \in \text{Sing}(\mathbb{E}(\pi_\eta)).$$

Now the relations

$$4 = h^0(\mathcal{O}(\tilde{D}) \otimes \mathcal{O}(\tilde{p} + \tilde{q})) \geq h^0(D) + h^0(D \cdot \eta) + h^0(\mathcal{O}(\tilde{p} + \tilde{q})) = h^0(D) + 1 + 1$$

imply that  $h^0(D) = 2 = h^0(g_d^1)$ .

Finally if  $d = g - 3$  then for  $g = 7$ , if  $h^0(D) > 2$  then  $F$  has a  $g_4^r$  with  $r \geq 2$ . This by Clifford's theorem and non-hyper ellipticity of  $F$  is a contradiction.

Consider moreover that since  $h^0(K_F - 2\mathcal{O}(g_d^1)) \geq 1$ , the line bundle  $\mathcal{L} := K_F - 2\mathcal{O}(g_d^1)$  is a line bundle satisfying  $2\mathcal{O}(g_d^1) \otimes \mathcal{L} = K_F$ .  $\square$

**Proposition 3.2.** *Assume that  $F$  and the assumptions about it are as in Theorem 3.1. Let  $Q$  be a quadric of rank equal or less than 4 containing  $F_\eta$  such that one of its rulings cuts a complete  $g_d^1$  on  $F_\eta$  with  $g - 2 \leq d \leq g - 1$  and  $2\mathcal{O}(g_d^1) \otimes \mathcal{L} = K_F$ , for some line bundle  $\mathcal{L}$  which  $\mathcal{L}$  is of degree 0 or 2 on  $F$ . Then  $Q$  is a projectivized tangent cone of  $\text{Sing}(\mathbb{E}(\pi_\eta))$ .*

*Proof.* Assume that  $Q \in |\mathcal{I}_{F_\eta}(2)|$  is of rank equal or less than 4 such that one of its rulings cuts a complete  $g_d^1$  with  $g - 2 \leq d \leq g - 1$  and  $2\mathcal{O}(g_d^1) \otimes \mathcal{L} = K_F$ . If  $\mathcal{L} = \mathcal{O}_F(p_1 + p_2 + \cdots + p_t)$  and  $\pi_\eta^*(\mathcal{L}) = \mathcal{O}_{\tilde{F}}(\bar{p}_1 + \bar{p}_2 + \cdots + \bar{p}_t + \bar{q}_1 + \bar{q}_2 + \cdots + \bar{q}_t)$ , where  $\bar{p}_i$  and  $\bar{q}_i$  are conjugate, then for a sub divisor  $\tilde{D}_1$  of  $\tilde{D} = \bar{p}_1 + \bar{p}_2 + \cdots + \bar{p}_t + \bar{q}_1 + \bar{q}_2 + \cdots + \bar{q}_t$  which is of degree  $\frac{1}{2} \deg(\tilde{D})$  and no two points of its support are conjugate, setting

$$\frac{1}{2}(\pi_\eta^* \mathcal{L}) := \mathcal{O}_{\tilde{F}}(\tilde{D}_1) \quad , \quad \tilde{\mathcal{L}} = \pi_\eta^*(\mathcal{O}(g_d^1)) \otimes \frac{1}{2}(\pi_\eta^* \mathcal{L}).$$

one has  $\text{Nm}(\tilde{\mathcal{L}}) = K_F$ . This reads to say that  $\tilde{\mathcal{L}} \in \text{Ker}(\text{Nm}) = \mathbb{P}(\pi_\eta) \cup Z_1$  where  $Z_1$  is the isomorphic copy of  $\mathbb{P}(\pi_\eta)$  which we already introduced in backgrounds.

Consider the relations:

$$\begin{aligned} h^0(\tilde{\mathcal{L}}) &= h^0(\pi_\eta^*(\mathcal{O}(g_d^1)) + \frac{1}{2}(\pi_\eta^* \mathcal{L})) \geq h^0(\pi_\eta^*(\mathcal{O}(g_d^1))) + h^0(\frac{1}{2}(\pi_\eta^* \mathcal{L})) \\ &= h^0(\mathcal{O}(g_d^1)) + h^0(\mathcal{O}(g_d^1) \cdot \eta) + h^0(\frac{1}{2}(\pi_\eta^* \mathcal{L})). \end{aligned}$$

If  $d = g - 1$  then  $h^0(\mathcal{O}(g_d^1)) = h^0(\mathcal{O}(g_d^1) \cdot \eta) = 2$  and  $\mathcal{L}$  has to be equal to  $\mathcal{O}_F$ . Therefore  $h^0(\tilde{\mathcal{L}}) = 4$  and so  $\tilde{\mathcal{L}} \in \text{Sing}(\mathbb{E}(\pi_\eta))$ . Moreover  $Q_{\tilde{\mathcal{L}}} = Q$  and this implies that  $Q$  is a projectivized tangent cone in this case.

In the case of  $d = g - 2$  one has  $h^0(\mathcal{O}(g_d^1) \cdot \eta) = 1$  which implies that

$$\begin{aligned} \dim(|\pi_\eta^*(\mathcal{O}(g_d^1))|) &= h^0(\pi_\eta^*(\mathcal{O}(g_d^1))) - 1 \\ &= h^0(\mathcal{O}(g_d^1)) + h^0(\mathcal{O}(g_d^1) \cdot \eta) - 1 = 2. \end{aligned}$$

This reads to say that taking a global section  $\sigma$  of  $\pi_\eta^*(\mathcal{O}(g_d^1))$  and considering its associated divisor,  $\tilde{D}$ , one has  $\dim(|\tilde{D}|) = 2$ . From this it can be seen that taking a global section  $\gamma$  of  $\tilde{\mathcal{L}}$  and considering its associated divisor  $\tilde{B}$ , there exists a divisor  $\tilde{E} \in |\tilde{B}|$  such that  $h^0(\tilde{E}) = 4$ . In fact for a point  $p$  in the support of a divisor associated to a global section of the line bundle  $\frac{1}{2}(\pi_\eta^*\mathcal{L})$ , assume that  $\tilde{D}_1 \in |\tilde{D}|$  is a divisor such that  $p \in \text{Supp}(\tilde{D}_1)$ . Now set  $\bar{D}_1 := \tilde{D}_1 + p + q = \bar{M} + 2p + q$  for some divisor  $\bar{M}$  on  $\tilde{F}$  and some point  $q \in \tilde{F}$  such that  $p + q \in \frac{1}{2}(\pi_\eta^*\mathcal{L})$ . Consider that  $\mathcal{O}(\bar{D}_1) \in |\tilde{\mathcal{L}}|$  and

$$\begin{aligned} \dim(<\bar{D}_1>) &= \dim(<\bar{M} + p + q>) \\ &= \dim(<\bar{M} + p>) + 1 = 2g - 6. \end{aligned}$$

This equivalently reads to say that  $h^0(\bar{D}_1) = 4$  which implies that  $h^0(\tilde{\mathcal{L}}) = 4$ . Therefore  $\tilde{\mathcal{L}} \in \text{Sing}(\mathbb{E}(\pi_\eta))$  which finishes the proof.  $\square$

#### 4. 2-NORMALITY OF GENERAL TETRAGONAL CURVES AND AN EXAMPLE OF PRYM TETRAGONAL CURVE

Let  $X$  be an irreducible plane sextic curve with  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  as its nodes or double points. Assume moreover that  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  are collinear. Let  $i : C \rightarrow X$  be its normalization. Notice that by genus formula for plane curves,  $X$ , and consequently  $C$ , is of genus 7. Now on the curve  $C$  consider the linear series  $|5H - \Delta|$  for which  $H = i^*(\mathcal{O}_X(1))$ ,  $\Delta = x_1 + x_2 + y_1 + y_2 + z_1 + z_2$  with  $i^{-1}(\bar{x}) = \{x_1, x_2\}$ ,  $i^{-1}(\bar{y}) = \{y_1, y_2\}$  and  $i^{-1}(\bar{z}) = \{z_1, z_2\}$ . Notice that  $K_C = 3H - \Delta$  and  $\deg(5H - \Delta) = 24$ . Now take a divisor  $D_{12} \in C^{(12)}$  such that  $2D_{12}$  belongs to  $|5H - \Delta|$  and set  $\eta := 2H - D_{12}$ . Trivially  $\eta^2 = 4H - 2D_{12} \sim 4H - (5H - \Delta) = \Delta - H \sim O$  where the last equality holds because the points  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  are collinear. Notice moreover that the lines passing through one of the singularities of the curve  $X$  define a base point free  $g_4^1$  on  $X$  and  $i^*(g_4^1)$  is a base point free  $g_4^1$  on  $C$ . This implies that  $C$  is an irreducible, nonsingular, tetragonal curve of genus 7 with three number of base point free  $g_4^1$ 's.

Next we verify the existence of a curve  $C$  admitting a  $D_{12}$  with mentioned properties:

**Theorem 4.1.** *There exists a curve  $C$  admitting a divisor  $D_{12}$  with the mentioned properties and admitting a globally generated and very ample prym-canonical line bundle  $K_C \cdot \eta$ .*

*Proof.* Let  $Q_1$  and  $Q_2$  be quadrics in  $\mathbf{P}^2$  tangent to each other exactly in one point. Bezout's theorem implies that they cut each other in two



extra points  $\bar{x}$  and  $\bar{y}$  with multiplicity one. Consider the line  $l$  passing through  $\bar{x}$  and  $\bar{y}$ . Since the tangent variety of  $Q_1$ , (resp.  $Q_2$ ) fills up all the surface  $\mathbf{P}^2$ , an arbitrary point  $\bar{z} \in l$  lies on at least a tangent line of  $Q_1$ , (resp.  $Q_2$ ). Therefore for an arbitrary point  $\bar{z}$  on  $l$  there are a couple of lines  $L_1$  and  $L_2$  passing through  $\bar{z}$  such that  $L_1$  is tangent to  $Q_1$  and  $L_2$  is tangent to  $Q_2$ . Then with this assignments, the reducible curve  $X$  defined by the polynomial  $h = Q_1Q_2L_1L_2$  is a curve of degree six which has three collinear ordinary singularities.

Denote by  $t$  the point where  $Q_1$  and  $Q_2$  are tangent to each other. A computation shows that there are infinitely many quadrics through  $\bar{x}$ ,  $\bar{y}$  and  $t$  such that each of these quadrics has the same tangent line at the point  $t$ . In fact, quadrics in  $\mathbf{P}^2$  passing through the points  $p = (1 : 0 : 0)$ ,  $q = (0 : 1 : 0)$  and  $r = (0 : 0 : 1)$ , are given by  $b_0x_0x_1 + b_1x_0x_2 + b_2x_1x_2 = 0$ . These quadrics have the line  $b_0x_0 + b_1x_2 = 0$  as their tangent line at the point  $p$ . These imply that for fixed  $d_0, d_2$  the infinitely many quadrics  $b_0x_0x_1 + b_1x_0x_2 + b_2x_1x_2 = 0$  pass through  $p, q, r$  and are tangent to each other at the point  $p$ .

Choose a couple of quadrics  $\bar{Q}_1, \bar{Q}_2$  passing through  $\bar{x}, \bar{y}, t$ , tangent to each other at  $t$  and distinct with  $Q_1$  and  $Q_2$  respectively. As in the sextic  $h$ , there are lines  $\bar{L}_1$  and  $\bar{L}_2$  distinct from  $L_1$  and  $L_2$  respectively, passing through  $\bar{z}$  and are tangent to  $\bar{Q}_1$  and  $\bar{Q}_2$  respectively. Again the curve  $k = \bar{Q}_1\bar{Q}_2\bar{L}_1\bar{L}_2$  is a reducible plane sextic having three collinear points  $\bar{x}, \bar{y}$  and  $\bar{z}$  as its ordinary singularities. Now since  $h$  and  $k$  has no common irreducible component, Bertini's theorem implies that  $X$ , a general member of the pencil generated by  $h$  and  $k$ , is an irreducible plane sextic having three collinear points as its ordinary singularities. Choosing the normalization of  $X$  gives the desired curve  $C$ .

To show existence of a  $D_{12}$  with desirable properties, take a plane quintic  $T$  with three number of nodes  $p_1, p_2, p_3$  and passing through three distinct prescribed collinear points  $\bar{x}, \bar{y}$  and  $\bar{z}$ . Choose nine extra points  $p_4, \dots, p_{12}$  on  $T$ . Notice that passing through points  $p_1, \dots, p_{12}$ , being tangent to  $T$  at the points  $p_4, \dots, p_{12}$  and having three collinear points  $\bar{x}, \bar{y}, \bar{z}$  as only singularities, impose at most 24 conditions on the space of plane sextics. Since the space of plane sextics is of dimension 27, there are plane sextics  $X$ , passing through the points  $p_1, \dots, p_{12}$ , being tangent to  $T$  at the points  $p_4, \dots, p_{12}$  and having three collinear points  $\bar{x}, \bar{y}, \bar{z}$  as only singularities. On such a curve  $X$ , setting  $X_{12} := p_1 + \dots + p_{12}$  one obtains the desired  $D_{12}$ .

It can be seen easily that any Prym-canonical line bundle on a non-hyperelliptic curve is globally generated. See [11, Lemma 2.1].

To see very ampleness of a prym-canonical line bundle on  $C$ , consider that the prym-canonical line bundle  $K_C \cdot \eta$  with  $\eta = 2H - D_{12}$  is very



ample. In fact the proof of Lemma 4.2 implies that in lack of very ampleness of  $K_C \cdot \eta$ , there would exist two another points  $z, w \in X$  such that

$$x + y \approx z + w \quad , \quad 2x + 2y \sim 2z + 2w \in g_4^1.$$

This means that taking  $L$ , the line passing through singularities of  $X$ , there exists a line  $\acute{L}$  other than  $L$  that is tangent to  $X$  in two points  $\alpha$  and  $\beta$  distinct with  $x, y$  and  $z$ . But this is impossible for a general curve of type  $X$ .  $\square$

More than what we saw in Theorem 4.1 and at least as an independent interest, one can prove that any Prym-canonical line bundle on a general tetragonal curve is very ample and the Prym-canonical model of this line bundle is projectively normal:

Assume for a moment that the line bundle  $K_X \cdot \eta$  is very ample. Then  $X_\eta$ , the Prym-Canonical model of  $X$  in  $\mathbf{P}(H^0(K_X \cdot \eta))$ , is 2-normal, namely the map

$$H^0(\mathbf{P}(H^0(K_X \cdot \eta)), \mathcal{O}_{\mathbf{P}(H^0(K_X \cdot \eta))}(2)) \rightarrow H^0(X_\eta, \mathcal{O}_{X_\eta}(2))$$

is surjective. In fact by [9], a curve  $X$  is  $n$ -regular if and only if, it is  $(n-1)$ -normal and the line bundle  $\mathcal{O}_X(n-2)$  is non-special. Therefore to prove 2-normality of  $X_\eta$ , it is enough to prove its 3-regularity, which means that its sheaf of ideals,  $\mathcal{I}_{X_\eta}$ , is 3-regular. Moreover for  $n \geq 1$ , using the exact sequence

$$0 \rightarrow \mathcal{I}_{X_\eta} \rightarrow \mathcal{O}_{\mathbf{P}(H^0(K_X \cdot \eta))} \rightarrow \mathcal{O}_{X_\eta} \rightarrow 0,$$

it is enough to prove that the sheaf  $\mathcal{O}_{X_\eta}$  is 2-regular. Trivially  $H^i(X_\eta, \mathcal{O}_{X_\eta}(2-i)) = 0$  for  $i \geq 2$  by Grothendieck's vanishing theorem. It remains to prove that  $H^1(X_\eta, \mathcal{O}_{X_\eta}(1)) = 0$ . To see this, consider the isomorphisms

$$H^1(X_\eta, \mathcal{O}_{X_\eta}(1)) \cong H^1(X, K_X \cdot \eta) \cong (H^0(X, K_X - K_X \cdot \eta))^\vee = (H^0(X, \eta))^\vee$$

together with  $H^0(X, \eta) = 0$ . These finish, 2-normality of  $X_\eta$ .

The discussion just has been done, proves projective normality of curves  $X$  with  $\text{Cliff}(X) \geq 3$ , in which case any prym-canonical line bundle  $K_X \cdot \eta$  is very ample by [11]. In the case  $\text{Cliff}(X) = 2$ , it can happen that the line bundle  $K_X \cdot \eta$  is not very ample for special tetragonal curves  $X$ . But it can be proved that for a general tetragonal curve  $X$ , any prym-canonical line bundle  $K_X \cdot \eta$  is very ample. In fact, an equality

$$h^0(K_X \cdot \eta(-x-y)) = h^0(K_X \cdot \eta) - 1$$

for some points  $x, y$  on  $X$ , implies that there exist another points  $z, w \in X$  such that

$$x + y \approx z + w \quad , \quad 2x + 2y \sim 2z + 2w \in g_4^1.$$

This by [6] is absurd for a general tetragonal curve.  
Summarizing we have proved:

**Theorem 4.2.** *Assume that  $X$  is a general smooth tetragonal curve of genus  $g$  and  $\tilde{X} \rightarrow X$  an étale double covering of  $X$  induced by  $\sigma \in \text{Pic}(X)$  with  $\sigma^2 = 0$ . Then  $X_\sigma$ , the Prym-canonical model of  $X$  in  $\mathbf{P}(H^0(K_X \cdot \sigma))$ , is projectively normal in  $\mathbf{P}(H^0(K_X \cdot \sigma))$ .*

## 5. PROJECTIVIZED TANGENT CONE QUADRICS OF $C$ GENERATE THE SPACE OF $H^0(\mathcal{I}_{C_\eta}(2))$

Everywhere in this paper by  $C$  we mean the tetragonal curve obtained in Theorem 4.1. Moreover by  $\eta$  we mean the 2-torsion line bundle obtained there in the rest of the paper.

**Theorem 5.1.** *Let  $C$  and  $\eta$  be the curve and the line bundle obtained in Theorem 4.1. Assume moreover that the double covering induced by  $\eta$  is an étale. Then a quadric  $Q \in \mathcal{I}_{C_\eta}(2)$  of rank equal or less than 4 is a projectivized tangent cone if and only if one of its rulings cuts a complete  $g_d^1$  with  $d \in \{g-3, g-2, g-1\}$  and  $2\mathcal{O}(g_d^1) \otimes \mathcal{L} = K_C$ , for some line bundle  $\mathcal{L}$  on  $C$  which is of degree 0, 2 or 4.*

*Proof.* If a quadric  $Q \in \mathcal{I}_{C_\eta}(2)$  is a projectivized tangent cone of  $\mathbb{E}(\pi_\eta)$ , then since  $g(C) = 7$  one of its rulings cuts the desired complete linear series, by Theorem 3.1.

Conversely if one of the rulings of a quadric  $Q \in \mathcal{I}_{C_\eta}(2)$  of rank equal or less than 4 cuts a complete  $g_d^1$  with prescribed conditions, then Proposition 3.2 implies that  $Q$  is a projectivized tangent cone of  $\mathbb{E}(\pi_\eta)$  provided that  $d \in \{g-2, g-1\}$ . If  $d = g-3$ , then one obtains a  $g_4^1$  on  $C$ . By Martens-Mumford's theorem there are only finitely many  $g_4^1$ 's on  $C$ . In fact the pencils of lines through  $\bar{x}$  through  $\bar{y}$  or through  $\bar{z}$  cut out three  $g_4^1$ 's on  $C$  and one can see that a  $g_4^1$  on  $C$  is one of these pencils. But it is easy to see that the rulings generated by divisors in these  $g_4^1$ 's are at most of dimension 2 and therefore these rulings can not sweep out a quadric. In fact for  $D \in |H - (x_1 + x_2)|$  one has:

$$\begin{aligned} \dim(< D >) &= 5 - h^0(K_C \cdot \eta - g_4^1) \\ &= 5 - h^0(3H - \Delta + 2H - D_{12} - (H - (x_1 + x_2))) \\ &= 5 - h^0(D_{12} - H + x_1 + x_2) \leq 5 - h^0(D_{12} - H). \end{aligned}$$

Since  $C$  is non-hyperelliptic, the Clifford's theorem asserts that  $h^0(D_{12} - H) \leq 3$ . Consider now that multiplying by  $H$  gives the following exact sequence:

$$0 \rightarrow H^0(D_{12} - H) \rightarrow H^0(D_{12}) \rightarrow H^0(H)$$

which implies that  $h^0(D_{12}) - h^0(D_{12} - H) \leq h^0(H) = 3$ . Therefore one has  $h^0(D_{12} - H) \geq 3$ . Summarizing one has  $h^0(D_{12} - H) = 3$  and so  $\dim(< D >) \leq 5 - h^0(D_{12} - H) = 2$ .

These imply that  $d$  can not be equal to  $g - 3 = 4$  because the linear spaces inside  $Q$  generated by divisors in  $g_d^1$  have to sweep out the quadric itself.  $\square$

Now, in order to give an application of Theorem 5.1, we describe  $W_5^1$  and  $W_6^1$  on  $C$ :

**Example 5.2. (i)  $g_5^1$ 's on  $C$ :**

By Mumford-Martens theorem and considering that  $C$  is not hyperelliptic nor trigonal and nor a smooth plane quintic, one has  $\dim(W_5^1) \leq 5 - 2 - 2 = 1$ . For each  $p \in X - \{\bar{x}, \bar{y}, \bar{z}\}$ , the lines passing through  $p$  cut out a pencil of degree 5 on  $X$  as well as do the quadrics through  $\bar{x}, \bar{y}, \bar{z}$  and  $p$ . These pencils give rise to pencils of the same kind on  $C$  via pulling them back to  $C$  by the normalization map,  $i$ . Consider moreover that the only way for a one dimensional sub vector space  $V$ , of quadrics in  $\mathbf{P}^2$  to cut a  $g_5^1$  on  $X$  is that each member of  $V$  has to pass through  $\bar{x}, \bar{y}, \bar{z}$  and  $p$ . The cubics in  $\mathbf{P}^2$  can not cut a  $g_5^1$  on  $X$ . Generally picking  $6d - 11$  points  $p_1, p_2, \dots, p_{6d-11}$  fixed on  $X$ , the hypersurfaces of degree  $d$  in  $\mathbf{P}^2$  passing through  $\bar{x}, \bar{y}, \bar{z}$  and the chosen  $6d - 11$  points  $p_1, p_2, \dots, p_{6d-11}$  will cut a  $g_5^1$  on  $X$ . For  $d \geq 4$  we have  $r \geq 2$  and therefore hypersurfaces of degree  $d \geq 4$  won't cut a  $g_5^1$ . Therefore  $\dim(W_5^1) = 1$  and  $g_5^1$ 's are cut on  $X$  by lines or quadrics of  $\mathbf{P}^2$ . Now these pencils pulled back via  $i$ , are the only  $g_5^1$ 's on  $C$ .

If  $g_5^1 = H - p$  for some  $p \in C$  then  $K_C \cdot \eta - g_5^1 = D_{12} - H + p$ . Consider that  $h^0(D_{12}) = 6$  and by proof of Proposition 3.2, one has  $h^0(D_{12} - H) = 3$ . Therefore  $h^0(D_{12} - H - p) = 3$  if  $p$  belongs to the base locus of  $|D_{12} - H|$ , and  $h^0(D_{12} - H - p) = 2$  otherwise. Taking the exact sequence

$$0 \rightarrow H^0(D_{12} - H - p) \rightarrow H^0(D_{12} - H) \rightarrow H^0(H) \rightarrow 0$$

it is routine to see that  $h^0(K_C \cdot \eta - g_5^1) = 3$  if  $p$  belongs to the base locus of  $|D_{12} - H|$  and  $h^0(K_C \cdot \eta - g_5^1) = 2$  otherwise. These imply that for each divisor  $D \in |g_5^1|$  one has  $\dim(< D >) = 2$  if  $p$  belongs to the base locus of  $|D_{12} - H|$ , and  $\dim(< D >) = 3$  otherwise. Moreover consider that since by  $3H - \Delta \sim 2H$ , one has  $K_C = 2g_5^1 \otimes \mathcal{O}(2p)$ . Now Proposition 3.2 implies that the ruled hypersurface  $\cup_{D \in g_5^1} < \dot{D} >$  is a prym projectivized tangent cone provided that  $p$  does not belong to the base locus of  $|D_{12} - H|$ .

In the case  $g_5^1 = 2H - 2\bar{x} - 2\bar{y} - 2\bar{z} - p$  we have  $K_C = 2g_5^1 \otimes \mathcal{O}(2p)$  and

$$\begin{aligned} K_C \cdot \eta - g_5^1 &= 3H - \Delta + 2H - D_{12} - (2H - 2\bar{x} - 2\bar{y} - 2\bar{z} - p) \\ &= 4H - \Delta - D_{12} + p \sim D_{12} - H + p. \end{aligned}$$

Therefore the situation is the same as in the case  $g_5^1 = H - p$ .

(ii)  $g_6^1$ 's on  $C$ : Again by Mumford-Martens theorem and considering that  $C$  is not hyperelliptic nor trigonal and nor a smooth plane quintic, one has  $\dim(W_6^1) \leq 6 - 2 - 2 = 2$ . The lines in  $\mathbf{P}^2$  cut a  $g_6^2$  on  $C$ . For each  $p, q \in X - \{\bar{x}, \bar{y}, \bar{z}\}$  the quadrics through  $p, q$  and through two of the points  $\bar{x}, \bar{y}, \bar{z}$  cut a  $g_6^1$  on  $X$ . Again these pencils are pulled back to  $g_6^1$ 's on  $C$  via the normalization map.

Now similarly as in (i), the only way for a one dimensional sub-vector space  $V$ , of quadrics in  $\mathbf{P}^2$  to cut a  $g_6^1$  on  $X$  is that each member of  $V$  has to pass through two points  $p, q$  and through two of the points  $\bar{x}, \bar{y}, \bar{z}$ . A computation similar for  $g_5^1$ 's case shows that these are the only  $g_6^1$ 's on  $X$ . Any  $g_6^1$  on  $C$  will be obtained by pulling back a  $g_6^1$  on  $X$  via  $i$ . If  $g_6^1 = 2H - 2\bar{x} - 2\bar{y} - p - q$  then

$$\begin{aligned} K_C \cdot \eta - g_6^1 &= 3H - \Delta + 2H - D_{12} - (2H - 2\bar{x} - 2\bar{y} - p - q) \\ &\sim D_{12} - 2H + 2\bar{x} + 2\bar{y} + p + q \sim D_{12} - H + p + q - 2\bar{z}. \end{aligned}$$

Therefore one has

$$\begin{aligned} h^0(K_C \cdot \eta - g_6^1) &= h^0(D_{12} - H + p + q - 2\bar{z}) \\ &= h^0(D_{12} - H - 2\bar{z}) = h^0(D_{12} - H) - 2 = 1. \end{aligned}$$

where the last equality is valid because  $2\bar{z}$  is not contained in the base locus of  $|D_{12} - H|$ . These computations imply that for each  $D \in |g_6^1|$  one has  $\dim(< D >) = 4$  and the union of linear spaces  $< D > \subset \mathbf{P}^5$ , when  $D$  varies in  $g_6^1$ , fill up all the space  $\mathbf{P}^5$  and therefore the line bundle  $g_6^1$  can not give a prym projectivized tangent cone.

(iii)  $g_5^2, g_6^2$ 's on  $C$ : The curve  $C$  can't admit any  $g_5^2$  and  $\dim(W_6^2) \leq 0$ . Off course the lines in  $\mathbf{P}^2$  cut a  $g_6^2$  on  $X$ . The quadrics passing through the points  $\bar{x}, \bar{y}$  and  $\bar{z}$  cut a  $g_6^2$ . This is nothing but the  $g_6^2$  cut by the lines in  $\mathbf{P}^2$ . Again any  $g_6^2$  on  $C$  will be obtained by pulling back a  $g_6^2$  on  $X$  via  $i$ , the normalization map.

As a byproduct of the computations just have been done, Theorem 3.1 and Proposition 3.2, one has the following:

**Theorem 5.3.** *The space of quadrics containing the  $K_C \cdot \eta$ -model of  $C$  is generated by Prym projectivized tangent cones at double points of  $\mathbb{E}(\pi_\eta)$ . Precisely the set of quadrics*

$$\{Q_{g_5^1} \mid g_5^1 \in W_5^1\}$$

which consists of a subset of projectivized tangent quadrics of rank equal or less than 4, generate the space of quadrics containing  $C_\eta$  in  $\mathbf{P}^5$ .

*Proof.* There exists a map  $\Phi$  defined by

$$\begin{aligned} \Phi : W_5^1 &\rightarrow \mathbf{P}(\mathcal{I}_2(C)) \cong \mathbf{P}^2 \\ \Phi(g_5^1) &= Q_{g_5^1} := \cup_{D \in g_5^1} \langle D \rangle \end{aligned}$$

Consider that  $\Phi$  is an embedding and by our computations, its image is contained in the locus of projectivized tangent cone quadrics inside  $\mathbf{P}(\mathcal{I}_2(C))$ . Moreover consider again by example 5.2 that  $W_5^1$  consists of two copies both are birational to  $C$  itself. Therefore the image of the map  $\Phi$  is non-degenerate. These imply that the linear span of projectivized tangent cones is  $\mathbf{P}(\mathcal{I}_2(C))$ .  $\square$

*Remark 5.4.* Based on Debarre's work in [7], the Prym-Torelli map

$$\mathcal{P} : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$$

is generically injective. This fails in the non-generic locus' because of well known reasons. In fact Donagi's tetragonal construction as well as generalized tetragonal construction introduced in [8] imply that, for an etale double covering  $\bar{Y} \rightarrow Y$ , of a tetragonal (generalized tetragonal) curve  $Y$ , there exist two another un-ramified double coverings having Prymians isomorphic to that of  $\bar{Y} \rightarrow Y$ . This implies non-injectivity of the Prym-Torelli map in the locus of double coverings of tetragonal (generalized tetragonal) curves in  $\mathcal{R}_g$ .

On the other hand as it has been noticed by H. Lange and E. Sernesi for the injectivity of the Torelli map in [11], an effective strategy to deal with the injectivity of the Prym-Torelli map seems to consist of two main steps. The first step is to show that for a given unramified double cover  $\pi : \bar{X} \rightarrow X$ , the projectivized tangent cones at double points of the Prym-theta divisor of the Prym variety generate the space of quadrics through the Prym-canonical model of the double covering. This step has been done by Debarre in [7] for curves varying in an open subset of  $\mathcal{R}_g$ , as we already noticed.

The second main step consists in proving that the quadrics through the Prym-canonical model of  $\pi : \bar{X} \rightarrow X$  in the projective space of the Prym-canonical differential forms, cut the Prym-canonical model. This step has been also proved not only for general Prym-canonical curves by Debarre in [7], but also for general tetragonal curves of genus at least 11 by him in [6]. Meanwhile, H. Lange and E. Sernesi have done this step for unramified double coverings of curves  $X$  with  $\text{Cliff}(X) \geq 3$  in [11].

Consider now that Debarre's work in [6], together with non-injectivity of Prym-Torelli map in tetragonal (generalized tetragonal) locus, imply that the projectivized tangent cones at double points of the Prym-theta divisor of the Prym variety of a general tetragonal curve of genus at least 11, or that of a generalized tetragonal curve, do not generate the space of quadrics through the Prym-canonical model of an un-ramified double coverings of such a curve.

These however won't give any information about validity or invalidity of the first step for an arbitrary tetragonal curve of genus  $g \leq 11$ , as it is concluded from our work that the first step remains valid for an etale double cover of the curve  $C$ . This as well proves that the quadrics through the Prym-canonical model of an arbitrary etale double covering of the curve  $C$ , does not cut its Prym-canonical model.

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